SIGNATURE ____________________ NAME ______________________

Student ID # ____________________

Physics 402
Fall 2022
Prof. Anlage
FINAL Exam
20 December, 2022 8:00 AM – 10:00 AM
Closed Book, NO Calculator Permitted, CLOSED NOTES

Point totals are given in brackets for each part of the question.

If you run out of room, continue writing on the back of the same page. If you do so, make a note on the front part of the page!
Note: You must solve the problem following the instructions given in the problem. Correct answers alone will not receive full credit.

Partial Credit:

→ Show Your Work! Answers written with no explanation will not receive full credit.

→ You can receive credit for describing the method you would use to solve a problem, even if you missed an earlier part.

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Problem	Credit	Max. Credit
1		25
2		25
3		25
4		25
TOTAL		100

Please write and sign the Honor Pledge below. "I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Infinite square well $\mathcal{H}^0 = \frac{p^2}{2m} + V(x)$ $\quad V(x) = \begin{cases} 0 \text{ for } 0 < x < a \\ \infty \text{ for x<0 and x>a} \end{cases}$ $\quad E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$

and $\psi^0(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \text{ for } 0 < x < a \\ 0 \text{ for x<0 and x>a} \end{cases}$ with $n = 1, 2, 3, \ldots$

Harmonic Oscillator $H^0 = \frac{p_x^2}{2m} + \frac{k}{2}x^2 = \left(a_+a_- + \frac{1}{2}\right)\hbar\omega$, $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$, $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, $E_n = (n + \frac{1}{2})\hbar\omega$; $n = 0, 1, 2, \ldots$; $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$; $p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$; $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$; $a_-\psi_n = \sqrt{n}\psi_{n-1}$

Hydrogen Atom $\mathrm{H}^0 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{(-e)(+e)}{4\pi\varepsilon_0 r}$, $n = 1, 2, 3, \ldots$, $\ell = 0, 1, \ldots, n-1$, $-\ell \le m \le \ell$, $\psi_{n,\ell,m}^0(r,\theta,\varphi) = Const_{n,\ell} e^{-\frac{r}{na}} \left(\frac{2r}{na}\right)^{\ell} L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right) Y_\ell^m(\theta,\phi)$; $\psi_{100}^0(r,\theta,\varphi) = \frac{2}{\sqrt{4\pi}\, a^{3/2}}\, e^{-r/a}$; $\psi_{200}^0(r,\theta,\varphi) = \frac{2}{\sqrt{4\pi}\,(2a)^{3/2}} \left(1 - r/(2a)\right) e^{-r/2a}$; $E_n = -13.6\ eV/n^2$, $L^2|\ell\ m_\ell\rangle = \ell(\ell+1)\hbar^2|\ell\ m_\ell\rangle$, $L_z|\ell\ m_\ell\rangle = m_\ell\hbar|\ell\ m_\ell\rangle$; $S^2|s\ m_s\rangle = s(s+1)\hbar^2|s\ m_s\rangle$, $S_z|s\ m_s\rangle = m_s\hbar|s\ m_s\rangle$; $J^2|j\ m_j\rangle = j(j+1)\hbar^2|j\ m_j\rangle$, $J_z|j\ m_j\rangle = m_j\hbar|j\ m_j\rangle$. $\vec{J} = \vec{L} + \vec{S}$ Code letters: "s" means $\ell = 0$, "p" means $\ell = 1$, "d" means $\ell = 2$, "f" means $\ell = 3$, etc. Term notation: ${}^{2S+1}L_J$

Spin-1/2 $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Singlet: $|0\ \ 0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\downarrow\rangle\right)$; triplet: $|1\ \ 1\rangle = |\uparrow\rangle|\uparrow\rangle$; $|1\ \ 0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle\right)$; $|1\ \ -1\rangle = |\downarrow\rangle|\downarrow\rangle$

$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, and $\hat{S}_\pm|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s \pm 1)}\ |s, m_s \pm 1\rangle$.

Perturbation theory $\mathrm{H}^0\psi_n^0 = E_n^0\psi_n^0$, $H = H^0 + H^1$, $\mathrm{H}\psi_n = E_n\psi_n$;

$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \ldots$; $E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots$; $E_n^1 = \int \psi_n^{0*}\ \mathrm{H}'\ \psi_n^0\ d^3r$

$\psi_n^1 = \sum_{\ell \ne n} \left(\frac{\int \psi_\ell^{0*}\ \mathrm{H}'\ \psi_n^0\ d^3r}{E_n^0 - E_\ell^0} \right) \psi_\ell^0$, $E_n^2 = \sum_{k \ne n} \frac{\left| \int \psi_k^{0*}\ \mathrm{H}'\ \psi_n^0\ d^3r \right|^2}{E_n^0 - E_k^0}$, Rel. correction to T: $\mathrm{H}' = -\frac{p^4}{8m^3c^2}$;

Relativistic energy correction: $E_{n,\ell}^1 = -|E_n^0| \frac{\alpha^2}{4n^2}\left[\frac{4n}{\ell + \frac{1}{2}} - 3\right]$; $\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} \cong \frac{1}{137.036}$;

$\overline{\overline{W}}\vec{\alpha} = E^1\vec{\alpha}$ $\quad W_{k,j} \equiv \langle\psi_k^0|H'|\psi_j^0\rangle$ $\quad \mathrm{H}_{so} = -\vec{\mu} \bullet \vec{B}$; $\vec{\mu} = -\frac{e}{m}\vec{S}$ for the electron;

$\vec{S} \bullet \vec{L} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$; Spin-orbit energy correction: $E_{n,\ell,s,j}^1 = \frac{|E_n^0|\alpha^2}{n} \frac{j(j+1) - \ell(\ell+1) - 3/4}{2\ell(\ell + \frac{1}{2})(\ell+1)}$;

$\Delta E = E_n^{1Relativity} + E_n^{1SpinOrbit} = \frac{|E_n^0|\alpha^2}{n^2}\left[\frac{3}{4} - \frac{n}{j + \frac{1}{2}}\right]$;

$\left|j\ m_j\right\rangle = \sum_{m_\ell + m_s = m_j} C_{m_\ell\ m_s\ m_j}^{\ell\ \ s\ \ j} \left|\ell\ m_\ell\right\rangle\left|s\ m_s\right\rangle$; $\vec{\mu}_{Total} = \vec{\mu}_\ell + \vec{\mu}_S = -\frac{e}{2m}(\vec{L} + 2\vec{S})$;

$\mathcal{H}_Z^1 = -\vec{\mu}_{Total} \cdot \vec{B}_{ext}$; $\quad E_Z^1 = \frac{e}{2m}\vec{B}_{ext} \cdot \langle\vec{J} + \vec{S}\rangle$; $\quad E_Z^1 = \mu_B g_J B_{ext} m_J$; $\quad \mu_B = \frac{e\hbar}{2m}$;

$g_J = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$; $E^1_{Z,strong} = \frac{e}{2m}\vec{B}_{ext} \cdot \langle \vec{L} + 2\vec{S} \rangle = \mu_B B_{ext}(m_\ell + 2m_s)$; $E^1_{fs} = \frac{|E^0_1|\alpha^2}{n^3}\left\{\frac{3}{4n} - \left[\frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]\right\}$;

$\mathrm{H}_{HF} = -\vec{\mu}_e \bullet \vec{B}_{dip}$; $E^1_{n,0,0} = \frac{\mu_0 g e^2}{3 m_e m_p} \frac{\hbar^2}{\pi n^3 a^3} \begin{cases} 1/4 & \text{TRIPLET} \\ -3/4 & \text{SINGLET} \end{cases}$ $\hat{P}\Psi(1,2) = \Psi(2,1)$

$\hat{P}^2 = 1$ $\Psi^0_A(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) - \psi_a(2)\psi_b(1)\right)$ $\Psi^0_S(1,2) = \frac{1}{\sqrt{2}}\left(\psi_a(1)\psi_b(2) + \psi_a(2)\psi_b(1)\right)$

$\langle (x_1 - x_2)^2 \rangle_\pm = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b \mp 2|\langle x \rangle_{ab}|^2$ with $\langle x \rangle_{ab} \equiv \int x\, \psi^*_a(x)\, \psi_b(x) dx$

Feynman-Hellmann theorem: $\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial \mathcal{H}}{\partial \lambda} | \psi_n \rangle$

Time-Dependent Perturbation Theory: $\mathcal{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$ $\Psi(\vec{r}, t) = \psi(\vec{r}) e^{-iEt/\hbar}$

$\dot{c}_a = -\frac{i}{\hbar}\mathcal{H}'_{ab} e^{-i\omega_0 t} c_b$, $\dot{c}_b = -\frac{i}{\hbar}\mathcal{H}'_{ba} e^{+i\omega_0 t} c_a$, $\mathcal{H}'_{ab} \equiv \langle \psi_a | \mathcal{H}' | \psi_b \rangle$, $\omega_0 = (\mathrm{E}_b - \mathrm{E}_a)/\hbar$.

Two-level system: $c_a(0) = 1$, $c_b(0) = 0$, $c_b(t) = -\frac{i}{\hbar}\int_0^t \mathcal{H}'_{ba}(t')\, e^{i\omega_0 t'} dt'$.

$\dot{a}_{nj} = \frac{-i}{\hbar} e^{i(E^0_j - E^0_n)t/\hbar} \int \psi^*_j(\vec{x}) \mathrm{H}'(\vec{x}, t) \psi_n(\vec{x}) d^3x$ with $|a_{nj}|^2$ the transition probability from state n to j. Sinusoidal perturbation: $\mathcal{H}'(\vec{r}, t) = V(\vec{r}) \cos \omega t$; $P_{a\to b}(t) = |c_b(t)|^2 \cong \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0 - \omega)t/2]}{(\omega_0 - \omega)^2}$ with $V_{ab} \equiv \langle \psi_a | V(\vec{r}) | \psi_b \rangle$; Spontaneous emission rate: $A = \frac{\omega_0^3 |\vec{\wp}|^2}{3\pi\varepsilon_0 \hbar c^3}$ with $|\vec{\wp}| \equiv q\langle \psi_b | \vec{r} | \psi_a \rangle$; $A = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{\pi e^2}{3\varepsilon_0 \hbar^2}\left(|x_{ab}|^2 + |y_{ab}|^2 + |z_{ab}|^2\right)$; Electric dipole selection rules: No transitions occur unless $\Delta m = \pm 1$ *or* 0 and $\Delta \ell = \pm 1$;

$$\begin{cases} \text{if } m' = m, & \text{then } \langle n'\ell'm' | x | n\ell m \rangle = \langle n'\ell'm' | y | n\ell m \rangle = 0 \\ \text{if } m' = m \pm 1, & \text{then } \langle n'\ell'm' | x | n\ell m \rangle = \pm i \langle n'\ell'm' | y | n\ell m \rangle \\ & \text{and } \langle n'\ell'm' | z | n\ell m \rangle = 0 \end{cases}$$

Planck blackbody radiation: $\rho(\omega) = \frac{\hbar\omega^3/(\pi^2 c^3)}{e^{\hbar\omega/k_B T} - 1}$; Absorption probability due to incoherent light: $R_{a\to b} = \frac{\pi e^2 \rho(\omega_{ab})}{3\varepsilon_0 \hbar^2}\left(|x_{ab}|^2 + |y_{ab}|^2 + |z_{ab}|^2\right) \equiv \rho(\omega_{ab}) M_{ab}$; Fermi's golden rule for transition from a discrete initial state 'i' to a final state in the continuum: $R_{i\to f} = \frac{2\pi}{\hbar} \left|\frac{\mathcal{H}_{if}'}{2}\right|^2 g(E_f)$, where $g(E_f)$ is the density of states at the final energy.

WKB semi-classical approximation: $\psi(x) = \frac{D}{\sqrt{p_{class}(x)}} \exp\left[\pm \frac{i}{\hbar}\int^x p_{class}(x')dx'\right]$;

$p_{class} = \sqrt{2m(E - V(x))}$; For infinite square well potentials: $\frac{1}{\hbar}\int_0^a \sqrt{2m(E_n - V(x))}\,dx = \pi n$, with $n = 1, 2, 3, ...$; For finite wells: $\int_{x_1}^{x_1} \sqrt{2m(E_n - V(x))} = \pi\hbar\left(n - \frac{1}{2}\right)$, with $n = 1, 2, 3, ...$ and x_1, x_2 the classical turning points; For tunneling:

$$\psi(x) = \frac{D}{\sqrt{|p_{class}(x)|}} \exp\left[\pm\frac{1}{\hbar}\int^{x}|p_{class}(x')|dx'\right]; \quad T \propto e^{-2\gamma}, \quad \text{where} \quad \gamma = \frac{1}{\hbar}\int_0^a |p_{class}(x')|dx';$$

Fowler-Nordheim tunneling: $T = exp\left[-\frac{4}{3}\frac{\sqrt{2m}}{\hbar}\frac{\Phi^{3/2}}{e\mathcal{E}}\right]$; Lifetime: $\tau = \frac{2r_1}{v}e^{2\gamma}$

Variational Method: $E_{GS} \le \langle \Psi_{GS,Guess} | \mathrm{H} | \Psi_{GS,Guess} \rangle$;

$$\frac{\partial\langle \Psi_{GS,Guess}(\vec{r},\lambda_1,\lambda_2,\lambda_3,\ldots)|\mathrm{H}|\Psi_{GS,Guess}(\vec{r},\lambda_1,\lambda_2,\lambda_3,\ldots)\rangle}{\partial\lambda_i} = 0$$

Quantum Scattering Theory: $N_{scatt}(into\ d\Omega\ around\ \theta,\varphi) = N_{inc}n_{target}\frac{d\sigma}{d\Omega}(\theta,\varphi)d\Omega$, where $\frac{d\sigma}{d\Omega}(\theta,\varphi)$ is the differential scattering cross section (DSCS). $\sigma = \iint \frac{d\sigma}{d\Omega}(\theta,\varphi)\, d\Omega$, $d\Omega = \sin\theta\, d\theta\ \ d\phi$, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$, b =impact parameter, Rutherford scattering: $\frac{d\sigma}{d\Omega} = \left(\frac{qQ/4\pi\varepsilon_0}{4E\sin^2(\theta/2)}\right)^2$. Quantum scattering $\psi(r,\theta) = A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}$, $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, Partial wave analysis: $\psi(r,\theta) = A\left\{e^{ikz} + k\ \sum_\ell i^\ell\ (2\ell+1)\ a_\ell\ h_\ell^{(1)}(kr)\ \ P_\ell(\cos\theta)\right\}$, $h_\ell^{(1)}(kr) = j_\ell(kr) + in_\ell(kr)$ and $h_\ell^{(2)}(kr) = j_\ell(kr) - in_\ell(kr)$, $h_\ell^{(1)}(kr \gg 1) \to \frac{e^{ikr}}{r}$, $\sigma = \iint D(\theta)\, d\Omega = \iint |f(\theta)|^2\, d\Omega = 4\pi\sum_{\ell=0}^{\infty}(2\ell+1)\,|a_\ell|^2$, $\psi(r,\theta) = A\left\{\sum_\ell i^\ell\ (2\ell+1)\left[j_\ell(kr) + i\ k\ a_\ell\ h_\ell^{(1)}(kr)\right]\ P_\ell(\cos\theta)\right\}$. Scattering phase shifts: $a_\ell = \frac{1}{2ik}\left(e^{i2\delta_\ell} - 1\right) = \frac{1}{k}e^{i\delta_\ell}\sin\delta_\ell$, $\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\ \sin^2(\delta_\ell)$. TISE re-written: $(\nabla^2 + k^2)\psi = Q$, with $E = \frac{\hbar^2k^2}{2m}$, and $Q \equiv \frac{2m}{\hbar^2}V\,\psi$, Green's function satisfies $(\nabla^2 + k^2)G = \delta^3(\vec{r})$, so that $\psi(\vec{r}) = \int G(\vec{r} - \vec{r_0})\, Q(\vec{r_0})\, d^3\vec{r_0}$, with $G(\vec{r}) = -\frac{e^{ikr}}{4\pi r}$. Lippmann-Schwinger equation: $\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int\frac{e^{ik|\vec{r}-\vec{r_0}|}}{|\vec{r}-\vec{r_0}|}\ V(\vec{r_0})\,\psi(\vec{r_0})\, d^3\vec{r_0}$, where $\psi_0(\vec{r})$ is a free-particle solution. Scattering function in the $|\vec{r}| \gg |\vec{r_0}|$ approximation: $f(\theta) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec{k}\cdot\vec{r_0}}\,V(\vec{r_0})\,\psi(\vec{r_0})\, d^3\vec{r_0}$, first Born approximation: $f(\theta) \cong -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r_0}}\,V(\vec{r_0})\, d^3\vec{r_0}$, $f_{Low-Energy}(\theta) \approx -\frac{m}{2\pi\hbar^2}\int V(\vec{r_0})\, d^3\vec{r_0}$, $f_{spherically\ Symmetric}(\theta) = -\frac{2m}{\hbar^2\,\kappa}\int_0^\infty r_0\, V(r_0)\sin(\kappa\, r_0)\ dr_0$, with $\vec{\kappa} \equiv \vec{k'} - \vec{k}$, and $\kappa = 2k\sin\left(\frac{\theta}{2}\right)$. Born series expansion $\psi = \psi_0 + \int g\,V\,\psi_0 + \iint g\,V\,g\,V\ \psi_0 + \iiint g\,V\,g\,V\,g\,V\,\psi_0 + \cdots$

Free-Electron Fermi Gas: 3D infinite square well ($L_x \times L_y \times L_z = V$) with periodic (Born-von Karmen) boundary conditions: $\psi(x,y,z) \sim \frac{1}{\sqrt{V}}\ e^{ik_xx}\ e^{ik_yy}\ e^{ik_zz}$ with $\vec{k} = 2\pi\left(\frac{n_x}{L_x},\frac{n_y}{L_y},\frac{n_z}{L_z}\right)$, $n_x = 0, \pm1, \pm2, \pm3, \ldots$, etc. $\vec{p} = -i\hbar\vec{\nabla}$ has eigenvalue $\hbar\vec{k} = \hbar(k_x,k_y,k_z)$, each state takes up a volume of $\frac{2\pi}{L_x}\times\frac{2\pi}{L_y}\times\frac{2\pi}{L_z} = \frac{(2\pi)^3}{V}$ in k-space. Fermi wavevector $k_F = (3\pi^2\rho)^{1/3}$, where $\rho \equiv Nq/V$ is the number density of electrons (q = valence of atoms), Fermi energy $E_F = \frac{\hbar^2k_F^2}{2m} = \frac{\hbar^2}{2m}(3\pi^2\rho)^{2/3}$. Total energy $U_{Total} =$

$\frac{\hbar^2}{2m}\frac{V}{\pi^2}\frac{1}{5}k_F^5$, degeneracy pressure $P = \frac{2}{3}\frac{U_{Total}}{V} = \frac{\hbar^2}{5m}(3\pi^2)^{2/3}\rho^{5/3}$. Density of states (DOS) for spin-1/2 Fermions in 3D: $D(E)dE = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}\,dE$.

Cooper pairing problem: $\left\{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1,\vec{r}_2)\right\}\Psi(1,2) = E\Psi(1,2)$, $\Psi(1,2) = \sum_{k>k_F} g_k \cos\left(\vec{k}\bullet\vec{r}\right)|00\rangle$, TISE becomes $\left(E - 2\varepsilon_k\right)g_k = \sum_{k'} g_{k'}V_{k,k'}$, with $\varepsilon_k = \hbar^2 k^2/2m$ and $V_{k,k'} \equiv \int d^3r\, V(r)\exp\left[i\left(\vec{k}-\vec{k}'\right)\bullet\vec{r}\right]$. With the attractive interaction $V_{k,k'} = \begin{cases} -V \text{ when } E_F \le \varepsilon_k \le E_F + \hbar\omega_c, \text{where } V > 0 \\ 0 \text{ when } \varepsilon_k > E_F + \hbar\omega_c \end{cases}$, one finds a bound state: $E \cong 2E_F - 2\hbar\omega_c e^{-2/(N(E_F)V)}$, where $N(E_F)$ is the DOS at E_F.

Electrons in a Periodic Potential: The Kronig-Penney Model: Periodic potential of finite square wells of width b, depth V_0 and periodicity a such that $V(x+a) = V(x)$. Bloch's theorem: $\psi(x) = e^{iqx}u(x)$, where q is a real number (crystal momentum) and $u(x)$ has the same periodicity as the potential. Translation operator: $\hat{T}(a)\psi(x) = \psi(x - a)$. with $\hat{T}(a) = e^{-ia\hat{p}/\hbar}$. For any operator $\hat{Q}$: $\hat{T}(a)^\dagger\hat{Q}(\hat{x},\hat{p})\hat{T}(a) = \hat{Q}(\hat{x}+a,\hat{p})$. For the Kronig-Penney Hamiltonian: $\hat{\mathcal{H}}(\hat{x},\hat{p})\psi(x) = E\,\psi(x)$ and $\hat{T}(a)\psi(x) = \lambda\,\psi(x) \equiv e^{-iqa}\psi(x)$. In the deep and narrow well approximation $\frac{-\beta^2 ba}{2}\frac{\sin(\alpha a)}{\alpha a} + \cos(\alpha a) = \cos(qa)$ with $\alpha = \sqrt{\frac{2mE}{\hbar^2}}$, $\beta = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$, and q the crystal momentum. The dispersion relation $E = E(q)$ shows bands and band gaps.

Superfluid ^{4}He and Bose-Einstein condensation: Identical Bosons in a box solved with standing wave boundary conditions: $\psi(x,y,z) = A\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{a}\right)\sin\left(\frac{n_z\pi z}{a}\right)$ with $n_x = 1, 2, 3, \ldots$, etc. Energy level occupation in equilibrium at temperature T with chemical potential μ: $n_s = \frac{g_s}{e^{(E_s-\mu)/k_BT} - 1}$, with $s = (n_x, n_y, n_z)$. Enforcing the number constraint yields $N = n_1(T) + 2\pi V\left(\frac{2m}{h^2}\right)^{3/2}(k_BT)^{3/2}\Gamma\left(\frac{3}{2}\right)f(\mu/k_BT)$ with $f(\mu/k_BT) \equiv \sum_{p=1}^{\infty}\frac{e^{p\mu/k_BT}}{p^{3/2}}$. Crisis temperature for Bose-Einstein condensation:

$$T_c = \left(\frac{N/V}{2\pi \left(\frac{2mk_B}{h^2} \right)^{3/2} \Gamma\left(\frac{3}{2}\right) 2.612} \right)^{2/3}$$

. Macroscopic quantum wavefunction $\psi(\vec{r}) = \sqrt{\rho_s(\vec{r})} e^{i\theta(\vec{r})}$

and quantized circulation $\kappa = \oint \vec{v}_s \bullet d\vec{l} = \frac{nh}{m}$, where $n = 0, \pm 1, \pm 2, \ldots$

Mathematical Formulas $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$ $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$

$\int x \sin(ax) dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$; $\int x \cos(ax)\, dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$

$\int_0^\infty x^n\, e^{-x/a} dx = n!\, a^{n+1}$ $\int_0^\infty x^{2n}\, e^{-x^2/a^2} dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$\int_0^\infty x^{2n+1}\, e^{-x^2/a^2} dx = \frac{n!}{2} a^{2n+2}$

Integration by parts: $\int_a^b f \frac{dg}{dx} dx = -\int_a^b \frac{df}{dx} g\, dx + fg|_a^b$

$\int \sin\theta \cos\theta\, d\theta = -\int \cos\theta\, d(\cos\theta)$; $\int_0^\pi \sin^2(nu)\sin(u) du = \frac{4n^2}{4n^2 - 1}$; $\int_0^\infty r^n\, e^{-r/a}\, dr = a^{n+1}\, n!$

Binomial expansion for $x \ll 1$: $(1+x)^n \cong 1 + nx + \frac{n(n-1)}{2}x^2$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad (x < 1)$ $e^{-x} = 1 - x + x^2/2! + \cdots \quad (x \ll 1)$

$\sin\theta = \theta - \theta^3/3! + \cdots \quad (\theta \ll 1)$ $\cos\theta = 1 - \theta^2/2! + \cdots \quad (\theta \ll 1)$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$